2022

MATHEMATICS — HONOURS

Paper : CC-1

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer all the following multiple choice questions. Each question has four possible answers, of which exactly one is correct. Choose the correct option and justify your answer : 2×10

(a) If $n \in \mathbb{N}$ and $y = \sin 2x + \cos 2x$, then $y_{4n}(0) =$

(i) 2^{4n-1} (ii) 2^{4n}

(iii) 2^{4n+1} (iv) None of these.

(b) If $l = \lim_{x\to\infty}\left[x - \sqrt[4]{(x^2-1)(x^2-4)}\right]$, then the value of l is

(i) -3 (ii) 0

(iii) 3 (iv) None of these.

(c) The envelope of the family of ellipses $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, whose area is πc^2(constant) is

(i) $2xy = c^2$ (ii) $4xy = c^2$

(iii) $xy = c^2$ (iv) $xy = 2c^2$.

(d) The radius of curvature at any point (r, θ) for the curve $r = a(1 - \cos\theta)$ is

(i) $\sqrt{2ar}$ (ii) $\frac{1}{3}\sqrt{2ar}$

(iii) $\frac{2}{3}\sqrt{ar}$ (iv) $\frac{2}{3}\sqrt{2ar}$.

(e) The angle through which the axes must be turned so that the equation $lx + my + n = 0$ $(l \neq 0)$ may reduce to the form $ax + b = 0$, is equal to

(i) $\tan^{-1}\left(\frac{l}{m}\right)$ (ii) $\tan^{-1}\left(\frac{m}{l}\right)$

(iii) $\tan^{-1}\left(\frac{a}{b}\right)$ (iv) $\tan^{-1}\left(\frac{b}{a}\right)$.

(f) If the plane $x + kz = 2$ intersect the surface $x^2 + y^2 - z^2 + 1 = 0$ and $k > 0$, then which one is true?

(i) $k \in (1, 2)$ (ii) $k \in (0, 2)$

(iii) $k \in (1, 3)$ (iv) $k \in (0, 1)$.

(g) Perpendiculars PL, PM, PN are drawn from the point $P(a, b, c)$ to the coordinate planes. The equation of the plane LMN is

(i) $x + y + z = 2abc$ (ii) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2$

(iii) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$ (iv) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$.

(h) The point on the conic $r=\frac{21}{5-2\cos\theta}$, which has the smallest radius vector, is

(i) $(3, \pi)$ (ii) $(\pi, 3)$

(iii) $\left(-1, \frac{\pi}{2}\right)$ (iv) $(0, \pi)$.

(i) A force $\vec{F}=3\hat{i}+2\hat{j}-4\hat{k}$ is applied at the point $(1, -1, 2)$. Then the magnitude of the moment of $\vec{F}$ about the point $(2, -1, 3)$ is equal to

(i) $\sqrt{29}$ (ii) $\sqrt{57}$

(iii) $\sqrt{14}$ (iv) $\sqrt{53}$.

(j) If $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{a}\times\vec{c}=\vec{b}$, then which one is correct?

(i) $\vec{b}=\vec{c}\neq\vec{0}$ (ii) $\vec{b}=\vec{c}=\vec{0}$

(iii) $\vec{a}=\vec{c}=\vec{0}$ (iv) $\vec{a}=\vec{b}=\vec{0}$.

2. Answer ***any three*** questions :

(a) Show that the curve $y = 3x^5 - 40x^3 + 3x - 20$ is concave upwards in $-2 < x < 0$ and $2 < x < \infty$, but concave downwards in $-\infty < x < -2$ and $0 < x < 2$ and at $x = -2, 0, 2$ there are points of inflection. 5

(b) If $y = \cosh(\sin^{-1}x)$, then show that $(1 - x^2)y_{n+2} - (2x + 1)xy_{n+1} - (x^2 + 1)y_n = 0$. Hence find the value of y_n at $x = 0$. 3+2

(c) Find the rectilinear asymptotes of $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0$. 5

(d) Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $a^n + b^n = c^n$. 5

(e) If $I_n = \int_0^{\frac{\pi}{2}} \sec^n x\, dx$, then show that $(n-1)I_n = 2^{\frac{n-2}{2}} + (n-2)I_{n-2}$.

Hence show that $I_5 = \frac{1}{8}\left\{7\sqrt{2}+3\log\left(\sqrt{2}+1\right)\right\}$. 3+2

3. Answer ***any four*** questions :

(a) Show that the locus of a point whose distance from the pole is equal to its distance from the straight line $r\cos\theta + K = 0$ is $2r\sin^2\frac{\theta}{2} = K$. 5

(b) Reduce the equation $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$ to its canonical form and determine the nature of the conic. 4+1

(c) A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which is fixed and the plane through P and perpendicular to OP meets the axes in A, B, C. If the planes through the middle points of OA, OB and OC and parallel to the coordinate planes meet in a point Q, then show that the locus of Q is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=2\left(\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}\right)$. 5

(d) A plane passes through a fixed point (α, β, γ) and cuts the coordinate axes in A, B, C. Prove that the locus of the centre of the sphere OABC is given by $\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=2$. 5

(e) Show that the normals of the ellipsoid $x^2 + 2y^2 + 3z^2 = 984$ at the points $(12, -18, 8)$ and $(-6, 18, -10)$ are coplanar. Also show that these normals lie on the plane $x + y + z = 2$. 3+2

(f) Find the equation of the tangent plane to the paraboloid $ax^2 + by^2 = 2z$, parallel to the plane $lx + my + nz = 0$. 5

(g) Prove that among all central conicoids the hyperboloid of one sheet is the only ruled surface. 5

4. Answer ***any two*** questions :

(a) (i) If the vectors $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors, then show that the vectors $\vec{b}\times\vec{c},\ \vec{c}\times\vec{a},\ \vec{a}\times\vec{b}$ are non-coplanar.

(ii) Find the vector $\vec{r}$ such that $\vec{a}\times\vec{r}=\vec{b}$ and $\vec{a}\cdot\vec{r}=k$, where k is a scalar and $\vec{a},\vec{b}$ are given proper vectors. 3+2

(b) If $\vec{\alpha} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{\beta} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ have their usual meanings, then find $\frac{d}{dt}\left(\vec{\alpha} \times \frac{d\vec{\beta}}{dt}\right)$ at $t = 2$. 5

(c) If $\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k}$ when $t = 2$ and $\vec{r}(t) = 4\hat{i} - 2\hat{j} + 3\hat{k}$, when $t = 3$, then show that

$$\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10$$ 5
